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Why? Why? Why?: Future Teachers Discover Mathematical Depth

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Future Teachers' Discovery of Mathematical Depth

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Abstract
Acquiring depth in mathematics is a circular process. Depth comes from asking questions and exploring; yet, being able to ask questions and knowing which questions to ask come from realizing that mathematics makes sense and that it has depth. This article discusses depth and beauty in mathematics and talks about the importance of providing opportunities to our future teachers to learn to ask questions in order to delve deeply into mathematics.

> *"The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics."*
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> --G.H. Hardy, *A Mathematician's Apology* (London 1941)

"Why? Why? Why?" is the most common question a young child asks. One of the joys, and at the same time, one of the biggest challenges of parenthood is answering multiple questions, one after another after another and after another. How much detail to include in the answer without either dishonesty or providing more details than the child is capable of assimilating? When to set up a situation in which the child explores the questions to discover his or her own answer? How to answer in a way that encourages more questions…thus, establishing a path towards lifelong curiosity and learning?

Across the country, institutions of higher education teach the course *Mathematics Concepts for Elementary School Teachers*. One goal of such a course is to bring the future teachers back to asking "Why? Why? Why?" to return them to a once very familiar activity of their childhood. Do they need to know the answers? Yes, but the questions themselves are just as important. Adults generally feel less

[1] I would like to thank my colleagues for their insights and for the wonderful conversations we have had about these issues.

inclined to ask good questions than they did as children, especially in mathematics. So returning to the questioning mind of a child is an essential part of mathematics teacher preparation.

Teachers affect the attitudes of hundreds of children. Specifically, they touch the mathematical psyche of these children at a very early age. Teachers can nurture or kill the innate curiosity young children bring with them into the elementary school classroom. And ultimately, they will help determine whether children see mathematics as a subject to be feared or revered. Do teachers need to know how to arrive at answers? Yes, certainly. But more importantly, they need to ask questions. They need to <u>know</u> to ask the questions. They need to know <u>what</u> questions to ask. They need to <u>want</u> to ask the questions. Questions are crucial to understanding the depth of mathematics, and thus, getting a glimpse into its beauty. Asking questions and knowing what questions to ask come from a realization that mathematics has tremendous depth and beauty.

Depth is a measure of a subject's intensity and fullness. In mathematics, depth is not necessarily determined by the level of difficulty. Mathematics is more complex than complicated. Depth in mathematics is about the underlying connections between concepts, the generalization of ideas, the power of seemingly simple results, and the fact that mathematical facts make sense—everything in elementary mathematics, for example, has a logical explanation. These connections and generalizations take our understanding of mathematics and of the world around us in new directions and to new conclusions. These conclusions, in turn, cause us to ask more questions and discover new linkages and connections.

> *"The essence of mathematics is not to make simple things complicated, but to make complicated things simple."* –S. Gudder

For example, let's begin with some questions about simple arithmetic – addition and multiplication. Many young students (and their teachers) place addition and multiplication into the same category. After all, multiplication is simply repeated addition…right? Many people believe "If you know how to add, then you know how to multiply!" Let us assume that we have two two-digit numbers that we intend to add together. How do we do it? We all know that it is correct to add the units digits and then add the tens digits. Why, then, is the algorithm for multiplication different? Why can't we just multiply the units digits and then multiply the tens digits? What would we do after we find those products? Would we then add the products together? Multiply them? Are there other algorithms that would work? Why do they work? And going back to the first question: Is multiplication really just repeated addition? (Consider $\sqrt{3} \times \sqrt{2}$, for example.)[2]

The ability and disposition for inquiry are particularly important for the kinds of tasks that teachers need to perform each day as they help their students through the process of attaining understanding of concepts in order to achieve depth. They need to become adept at creating activities to achieve certain goals. Since students grasp concepts in different ways and at different rates, helping individual students requires knowing several different paths to achieve the same result, being able to construct easier problems as well as extensions, and having the ability to encourage the students to ask their own questions. Teachers also need to anticipate student responses and evaluate them for correctness.[3] All these skills can be improved by knowing how to ask and by expecting good questions. Teachers need to ask questions to delve into the depth of mathematics in order to bring the subject alive for their young students.

Here is an example of the way some mathematical questions and concepts may be linked with others via questioning. This is a true story. A few months ago we had our backyard redone and the whole lawn re-sodded. Unfortunately, the contractor made a big mistake in his calculation of the grassy area and approximately one eighth of our lawn remained un-sodded for several weeks. The following questions came to mind: What is area? How did the contractor make his estimate? How could he make such a

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[2] Liping Ma gives some nice examples of the depth in elementary mathematics in her book *Knowing and Teaching Elementary Mathematics,* including a comparison of approaches to answering some of the questions in this paragraph by Chinese and American teachers.

[3]Deborah Ball et al have published many articles about Specialized Mathematical Knowledge for teaching.

mistake? How do we find area? For many of us the immediate answer is "length times width." But that only applies to very specific shapes. He certainly could not have used only this formula to determine how much sod would be needed to cover our irregularly-shaped lawn. How do we find the area of irregular shapes? How many different strategies could we use? Could we use a scale model of the lawn to find the area? Could we figure out the area of the lawn using string? Do we know any other formulas that we can throw at this problem? Why is the area of a rectangle length times width? What is the area of a parallelogram? A trapezoid? A pentagon? Is there an area formula that applies to all pentagons? Why do the area formulas that we learned work? Why is the area of a circle π times the square of its radius?

We all learned about area in elementary school. My teacher introduced some of the formulas when I was in fourth grade. Consider the following questions from the previous paragraph and answer them before reading on.

Could we figure out the area of the lawn using string? That is, encircle the lawn using a long string, then create a square with the string, and simply find its area? Would this process result in the correct area?

Could we use a scale model of the lawn to find the area? How would we use it? For example, if we drew a scale model of the lawn such that the lengths in the picture were $1/k^{th}$ of the lengths in the lawn, could we multiply by k to find the area of the lawn?

These ideas are very clever! However, as tempting as these strategies would be, the answers to the questions are not all yes. Consider, for example, a 2cm x 4cm rectangle. This rectangle has an area of $8cm^2$ and perimeter 12cm. If we instead create a longer, skinnier rectangle, 1cm x 5cm, the perimeter is still 12cm, while the area changes to $5cm^2$. In fact, any given perimeter can enclose an infinite number of regions of different areas. Take a piece of string and tie the two ends together. How small can you get the enclosed area to be? How large? What is the shape of the largest possible region that can be enclosed by the string?

For the second question, imagine two squares, a 1 by 1 square and a 5 by 5 square. (Do units matter?) The lengths in the second square are 5 times as large as the lengths in the first square. Yet, the area of the larger square is 25 times as large as the area of the 1 by 1 square, not 5 times as large. The ratio of the areas of two shapes that are similar is not the same as the ratio of two corresponding lengths. These facts are elementary, yet many of us were not exposed to asking these types of questions, and thus may have certain assumptions about their answers that are not correct. Our intuition fails us.

Note that in the second case, the case of the scale model, we can actually find the area of the lawn if we know the area of the scale model by multiplying the area of the scale model by k^2. However, in the case of the string, it is not possible to find the area of the lawn just by knowing its perimeter.[4]

This is DEPTH! It's like pulling a string on a sweater, and having the whole sweater unravel. By pulling we unravel the barrier that hides the root of the issue, leading us to new issues. The thread connects many different mathematical ideas that create a whole picture. Our teachers need this ability. They must feel comfortable with these questions to ask them of themselves and to encourage questions in class. At the university level, we must prepare these teachers for such experiences. Then they will pass their insights on to their young pupils.

Let's pose a few more seemingly obvious questions. Note that there are many, many apparently unrelated questions that are connected to each other. The number of questions seems overwhelming, but the answer to each individual question is not complicated, if approached one at a time. For example, the following questions have a thread running through them. Let's unravel the sweater:

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[4] There are some laser guided measuring tools used by craftsmen that can calculate the area of a region by measuring its perimeters. However, this tool takes into consideration the shape of the region.

Why are there 360º around a circle?[5] Why is the sum of the measures of the interior angles of a triangle 180º? What is the sum of the measures of the interior angles of a square? Will this sum remain the same if we enlarge the square? What happens if we distort the square? What is the sum of the measures of the interior angles of a quadrilateral? Of a pentagon? Of a hexagon? Of a polygon with *n* sides? Why? Does it matter if the polygon is convex or concave? Why is it that we only see certain shapes of tiles in our kitchens and bathrooms? How many different shapes of tiles can there be and why? Which shapes can cover our kitchen floor without leaving gaps or overlapping? That is, which shapes tessellate the plane? Why? How many regular polygons tessellate the plane? How do the answers to these questions apply to the art by M.C. Escher? What does the measure of an angle have to do with a soccer ball? If there is an infinite number of regular polygons, why are there only 5 regular polyhedra (the equivalent of a regular polygon in three dimensions)? How are all these questions related to each other?

Again, this is DEPTH!! It is the ability to ask relatively simple questions which lead us in wonderfully interesting interrelated ways. Our teachers must have the ability to engage in this questioning so that our children will have the opportunity they deserve to also engage in this process. Their exploration will increase their appreciation and, of course, their understanding of math. Knowing how to ask the questions when we encounter a mathematical concept and exploring its depth will help our students discover its beauty.

From a very young age, we are taught to appreciate the beauty of literature, of art, of music, of poetry. We have music appreciation classes, art appreciation classes (probably not enough in elementary schools these days)…we go to concert halls to enjoy symphonies, to art museums to study and gaze at the masterpieces and marvel at the creativity of the artists. Unfortunately, mathematics is not viewed in this

[5] Actually, this question is not a mathematical question—though it's still important to ask—it was a choice made by the Babylonians, related to the fact that the period of rotation of the earth around the sun is 365 days. Some "mathematical facts" are historical accidents, arbitrary choices made in the past, while others are necessary logical consequences of definitions and axioms. It is crucial to understand this distinction and which facts belong to which category.

way. It is seen as a necessity…a gatekeeper for achieving career and educational success in many fields. And it is. In this technologically advancing society, a good mathematical foundation is becoming crucial for a large number of careers in science, engineering, medicine, technology, and even business, as an article in *Business Week* "Why Math Will Rock Your World" (January 23, 2006) recently pointed out. However, the applications of mathematics used in those fields, as well as mathematics, in general, are so much more than just skills. In the words of Richard Feynman, the Nobel Prize recipient in physics for 1965:

> *"To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature ... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in."*

Mathematics is more than just the steps needed to solve a problem. Mathematics is more than a set of isolated rules and procedures. After all, if mathematics were just a set of rules and regulations, would we mathematicians waste our time exploring it? Would we be so enthralled by it? Find it fascinating? It is the depth, the beauty of mathematics that attracts so many of us.

> *"All mathematicians share ... a sense of amazement over the infinite depth and the mysterious beauty and usefulness of mathematics."* -- Martin Gardner

But it can be more difficult to get a glimpse into the beauty of mathematics—many of us were trained in rote mathematics without the opportunity to appreciate its depth, and thus, its beauty. Yes, there are rules. There are procedures. But these rules and procedures make sense. They are interrelated, and deep beneath there is a web that connects them. It is the simple elegance of the structure underneath that makes mathematics beautiful. Reforms in school mathematics are challenging the ways in which math was traditionally taught—one of the goals is to understand the process, to know more than just algorithms. Mathematics requires more work as it gets more abstract—but understanding can be attained. As one of my students said in a letter to future students: "Get ready for an intense math experience. You are going to take everything you know about math and bring it to the next level…I now feel a lot more confident in my knowledge of math because I have the background knowledge and the proofs of everything I've learned. I know the *why* and not just the *how* of the math I do."

Our future teachers are very motivated to do anything that will benefit their future students. We can help the teachers become comfortable asking questions to explore the depth of mathematics and see its beauty. Then they will be more likely to spend the time required to grasp the concepts, and, once they understand the concepts, the procedures, with practice, will all fall into place. When teachers develop their inquiry abilities, they will be more motivated to persevere when stuck with a problem, as they will be able to ask the questions needed to break the problem down into bite-size pieces or to approach the problem in a different way. They may be surprised at how much they can actually understand if they continue to try, fighting the instinct to give up, even after experiencing frustration. And then, our teachers will be better able to help their own students acquire those skills and dispositions.

Let the melodies of mathematics resonate in our soul! Let us seek its depth, its beauty. Let our future teachers recall their own innate, useful wonder and curiosity, and their tenacity in searching for answers. Then they and their future students will ask not just "how?" in mathematics, but "Why? Why? Why?"