Bounds on the Number of Elastic Collisions in D-Dimensional Space

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Abstract

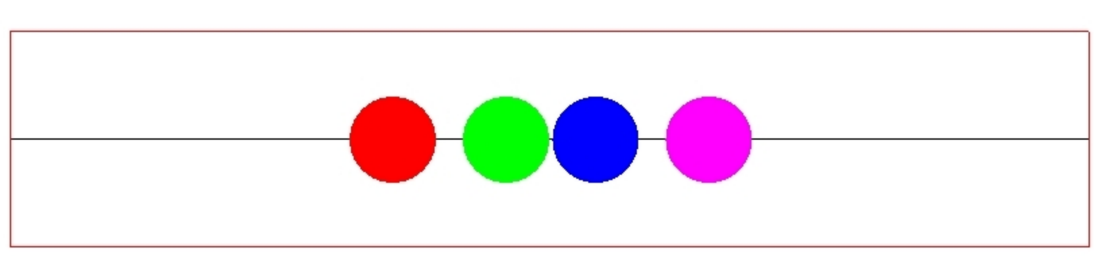
This interdisciplinary project focuses on improving the lower bound for the number of collisions of a finite system of n-balls in d-dimensional space. We developed software that computes all possible collisions between a system of balls with given initial positions and velocities, including collisions in positive and negative time. Building on the computation of collisions, we analyze various configurations of balls and their velocities in order to find configurations that produced more collisions than others.

Goals

- Create a visual representation of the collisions between a system of balls.
- Improve the lower bound for the maximum number of collisions for a system of balls.
- Model collisions between particles in fluids and gases.

Collisions in 1D

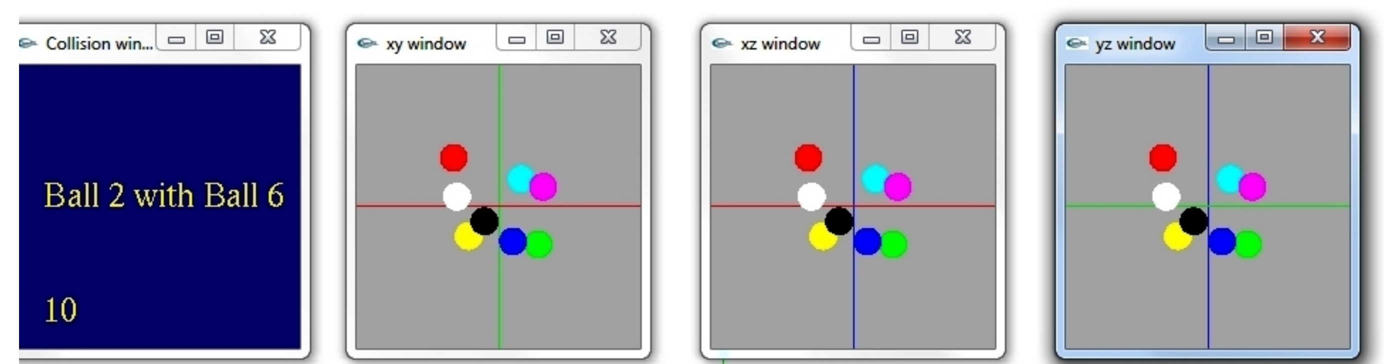
There is a huge gap between the upper and lower bounds for the maximum number of collisions between a system of n balls. However, in one-dimensional space the maximum number of collisions is known to be $n(n-1)/2$

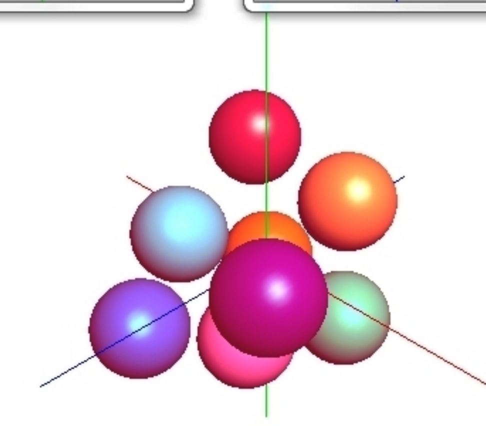


Case of c = 8 for n = 4, d = 3

Simulation and Heuristics

As an example, let's take the case of n=4 and d=3 (see pictures at the bottom of the poster). The positions and velocities of four balls in three-dimensional space can be determined by 4*3 + 4*3 = 24 numbers (three components of position and three of velocity for each ball). The basic idea of the so-called brute-force simulation approach is to select the input set - making sure that the balls do not penetrate each other initially - and following the laws of physics to compute the number of resulting collisions. Then repeat the experiment many, many times with different input sets.





A three-stage approach to simulation: In the first phase, random sets of 24 parameters are generated to find the sets that produce most collisions. Next, the more "promising" regions of the problem space are searched, which usually yields an increase in the number of collisions. Finally, various heuristics are used to perturb the input. I have developed several search heuristics of which one proved particularly successful (see below).

Strategies/ Results

Taking a system of balls in D-dimensional space, we can increase the dimensionality by adding a randomized component of velocity to one or more of the balls in the original system of balls. Increasing the dimensionality provides more opportunity for a greater number of collisions between the balls. Applying this heuristic, after billions of simulation runs, yielded a case of 8 collisions for n = 4, d = 3. This is a new result that has never been published in literature before.

Non-Central Collisions

The program uses the model of non-central collisions to compute the trajectories of the balls after collisions as well as the changed velocities. The computations are based on two principles; conservation of momentum and conservation of energy. This gives the following system of equations:

$$\begin{cases} \sum_{k=1}^{n} m_k v_k = const \\ \sum_{k=1}^{n} \frac{m_k v_k^{2}}{2} = const \end{cases}$$

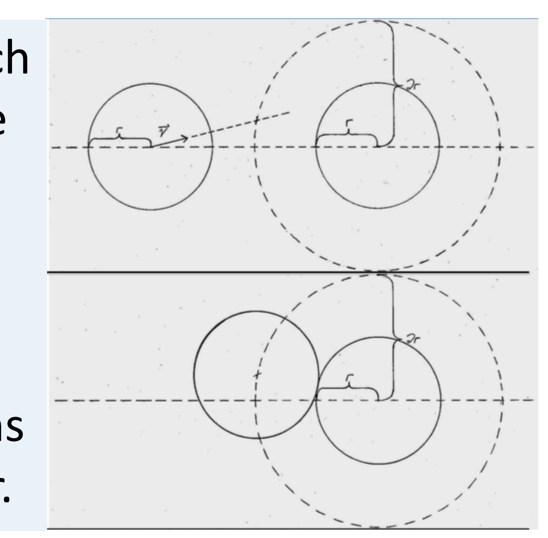
(k - ball number; momentum must be conserved in each dimension)

Collisions between each pair of balls are computed based on a method of collision detection used in computer graphics. Multidimensional calculus yields:

$$\begin{cases} (x-x_1)^2+(y-y_1)^2+(z-z_1)^2 = 4R^2 \\ x = x_2 + vx_2 t, \quad y = y_2 + vy_2 t, \quad z = z_2 + vz_2 t \end{cases}$$

(example for the case of d = 3)

Collisions are computed between each pair of balls and the collision with the smallest time is selected for processing. New velocities are computed from the laws of physics shown in the upper system of equations and the search for collisions repeats until no more collisions occur.



References

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